

# ON SOME ALGEBRAIC IDENTITIES AND THE EXTERIOR PRODUCT OF DOUBLE FORMS

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**ABSTRACT.** We use the exterior product of double forms to re-formulate celebrated classical results of linear algebra about matrices namely Cayley-Hamilton theorem, Laplace expansion of the determinant, Newton identities and Jacobi's formula for the determinant. This new formalism is then used to naturally generalize the previous results to double forms.

In particular, we show that the Cayley-Hamilton theorem once applied to the second fundamental form of a hypersurface is equivalent to a linearized version of the Gauss-Bonnet theorem, and once its generalization is applied to the Riemann curvature tensor (seen as a  $(2, 2)$  double form) is an infinitesimal version of the general Gauss-Bonnet-Chern theorem. In addition to that, the general Cayley-Hamilton theorems generate several universal curvature identities in the sense of Gilkey-Park-Sekigawa. The extension of the classical Laplace expansion of the determinant to double forms is shown to lead to general Avez type formulas for all the Gauss-Bonnet curvatures.

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## 1. EUCLIDEAN INVARIANTS FOR BILINEAR FORMS

Let  $A$  be a square matrix with real entries of size  $n$ . Recall that the characteristic polynomial  $\chi_A(\lambda)$  of  $A$  is given by

$$(1) \quad \chi_A(\lambda) = \det(A - \lambda I) = (-1)^n \lambda^n + (-1)^{n-1} s_1(A) \lambda^{n-1} + \dots + s_n(A) = \sum_{i=0}^n (-1)^{n-i} s_i(A) \lambda^{n-i}.$$

Where  $s_1(A)$  is the trace of  $A$ ,  $s_n(A)$  is the determinant of  $A$  and the other coefficients  $s_k(A)$  are intermediate invariants of the matrix  $A$  that interpolate between the trace and the determinant, we shall call them here the  $s_k$  invariants of  $A$ .

Since similar matrices have the same characteristic polynomial, therefore they have as well the same  $s_k$  invariants. In particular, one can define these invariants in an invariant way for endomorphisms.

Let  $(V, g)$  be an Euclidean vector space of finite dimension  $n$  and  $h$  be a bilinear form on  $V$ . We denote by  $\bar{h}$  the linear operator on  $V$  that corresponds to  $h$  via the inner product  $g$ .

We define the  $s_k$  invariants of the bilinear form  $h$  to be those of the linear operator  $\bar{h}$ . Note that in contrast with  $s_k(\bar{h})$ , the invariants  $s_k(h)$  depends on the inner product  $g$ . In order to make this dependence explicit we shall use the exterior product of double forms [3].

Recall that for  $1 \leq k \leq n$ , the exterior product  $h^k = h \dots h$ , where the bilinear form  $h$  is seen here as a  $(1, 1)$ -double form, is a multilinear form determined by the determinant as follows:

$$(2) \quad h^k(x_1, \dots, x_k, y_1, \dots, y_k) = k! \det[h(x_i, y_j)].$$

In particular, we have

$$(3) \quad \det h = * \frac{h^n}{n!}.$$

Where  $*$  denotes the (double) Hodge star operator operating on double forms, see [3]. Consequently, using the binomial formula, the characteristic polynomial of  $h$  takes the form

$$\begin{aligned} \chi_h(\lambda) &= \det(h - \lambda g) = * \frac{(h - \lambda g)^n}{n!} \\ &= * \frac{1}{n!} \sum_{i=0}^n \binom{n}{i} h^i (-1)^{n-i} \lambda^{n-i} g^{n-i} \\ &= \frac{1}{n!} \sum_{i=0}^n n! (-1)^{n-i} \left( * \frac{g^{n-i} h^i}{(n-i)! i!} \right) \lambda^{n-i} \\ &= \sum_{i=0}^n (-1)^{n-i} s_i(h) \lambda^{n-i}. \end{aligned}$$

We have therefore proved the following simple formula for all the  $s_k$  invariants of  $h$ :

**Proposition 1.1.** *For each  $0 \leq k \leq n$ , the  $s_k$  invariant of  $h$  is given by*

$$(4) \quad s_k(h) = \frac{1}{k!(n-k)!} * (g^{n-k} h^k).$$

Where  $*$  denotes the (double) Hodge star operator operating on double forms, and the products  $g^{n-k}, h^k, g^{n-k} h^k$  are exterior products of double forms, where  $g$  and  $h$  are considered as  $(1, 1)$ -double forms.

In particular, the trace and determinant of  $h$  are given by

$$s_1(h) = * \left\{ \frac{g^{n-1}}{(n-1)!} h \right\} \quad \text{and} \quad s_n(h) = * \frac{h^n}{n!}.$$

Let us note here that for any orthonormal basis  $(e_i)$  of  $(V, g)$ ,  $s_k(h)$  coincides by definition with the  $s_k$  invariant of the matrix  $(h(e_i, e_j))$ . In particular,  $s_n(h)$  is the determinant of the matrix  $(h(e_i, e_j))$ . More generally, we have the following lemma:

**Lemma 1.2.** *Let  $(e_i)$ ,  $i = 1, 2, \dots, n$ , be an orthonormal basis of  $(V, g)$ ,  $h$  a bilinear form on  $V$  and  $k + r \leq n$ . Then for any subset  $\{i_1, i_2, \dots, i_{k+r}\}$  with  $k + r$  elements of  $\{1, 2, \dots, n\}$  we have*

$$(5) \quad g^k h^r (e_{i_1}, e_{i_2}, \dots, e_{i_{k+r}}, e_{i_1}, e_{i_2}, \dots, e_{i_{k+r}}) = (k + r)! s_r (h(e_{i_a}, e_{i_b})).$$

Where  $s_r (h(e_{i_a}, e_{i_b}))$  is the  $s_r$  invariant of the  $(k + r) \times (k + r)$  matrix  $(h(e_{i_a}, e_{i_b}))$ , for  $1 \leq a, b \leq k + r$ ,  $g$  is the inner product on  $V$ . The product  $g^k h^r$  is the exterior product of double forms as above.

*Proof.* The characteristic equation of the  $(k + r) \times (k + r)$  matrix  $(h(e_{i_a}, e_{i_b}))$  is given by

$$\det (h(e_{i_a}, e_{i_b}) - \lambda I) = \det ((h - \lambda g)(e_{i_a}, e_{i_b})).$$

Consequently, formula (2) together with the binomial formula show that

$$\begin{aligned} \det (h(e_{i_a}, e_{i_b}) - \lambda I) &= \frac{1}{(k + r)!} (h - \lambda g)^{k+r} (e_{i_1}, e_{i_2}, \dots, e_{i_{k+r}}, e_{i_1}, e_{i_2}, \dots, e_{i_{k+r}}) \\ &= \frac{1}{(k + r)!} \sum_{i=0}^{k+r} (-1)^{k+r-i} \lambda^{k+r-i} g^{k+r-i} h^i (e_{i_1}, e_{i_2}, \dots, e_{i_{k+r}}, e_{i_1}, e_{i_2}, \dots, e_{i_{k+r}}). \end{aligned}$$

To complete the proof just recall that the  $s_r$  invariant of the  $(k + r) \times (k + r)$  matrix  $(h(e_{i_a}, e_{i_b}))$  is the coefficient of  $(-1)^k \lambda^k$  in the characteristic polynomial  $\det (h(e_{i_a}, e_{i_b}) - \lambda I)$ , that is,  $\frac{1}{(k+r)!} g^k h^r (e_{i_1}, e_{i_2}, \dots, e_{i_{k+r}}, e_{i_1}, e_{i_2}, \dots, e_{i_{k+r}})$ .  $\square$

In what follows in this section we shall use this formalism of the  $s_k$  invariants to reformulate and then generalize celebrated classical identities for matrices namely Laplace expansion of the determinant, Cayley-Hamilton theorem, Jacobi's formula for the determinant and Newton identities.

**1.1. Adjugates of bilinear forms.** Recall that the adjugate (called also adjoint or cofactor matrix) of a square matrix of size  $n$  is a new matrix formed by all the cofactors of the original matrix. Where the  $(ij)$ -cofactor of a matrix is  $(-1)^{i+j}$  times the determinant of the  $(n - 1 \times n - 1)$  sub-matrix that is obtained by eliminating the  $i$ -th row and  $j$ -th column of the original matrix.

We are going to do here the same transformation but on a bilinear form instead of a matrix. Precisely, let  $(e_i)$  be an orthonormal basis of  $(V, g)$ , for each pair of indexes  $(i, j)$ , we define, the  $(ij)$ -cofactor of the bilinear form  $h$ , denoted  $t_{n-1}(h)(e_i, e_j)$ , to be  $(-1)^{i+j}$  multiplied by the determinant of the  $(n - 1 \times n - 1)$  sub-matrix that is obtained after removing the  $i$ -th row and  $j$ -th column from the matrix  $(h(e_i, e_j))$ . Then we can use bilinearity to extend  $t_{n-1}$  into a bilinear form defined on  $V$ . It is then natural to name the so obtained bilinear form  $t_k(h)$  as the adjugate of  $h$ .

The next proposition shows in particular that the bilinear form  $t_{n-1}(h)$  is well defined (that is it does not depend on choice of the orthonormal basis)

**Proposition 1.3.** *If  $*$  denotes the Hodge star operator operating on double forms then*

$$(6) \quad t_{n-1}(h) = \frac{1}{(n-1)!} * (h^{n-1}).$$

*Proof.* From the definition of the Hodge star operator once it is acting on double forms we have

$$(7) \quad \frac{1}{(n-1)!} * (h^{n-1})(e_i, e_j) = \frac{1}{(n-1)!} (h^{n-1})(*e_i, *e_j).$$

The last expression is by formula (2) exactly equal to  $(-1)^{i+j}$  multiplied by the determinant of the  $(n-1) \times (n-1)$  sub-matrix that is obtained from  $(h(e_i, e_j))$  by removing the  $i$ -th row and  $j$ -th column. This completes the proof of the proposition.  $\square$

We define now higher adjugates of  $h$  as follows

**Definition 1.1.** For  $0 \leq k \leq n-1$  we define the  $k$ -th adjugate of  $h$  to be the bilinear form given by

$$(8) \quad t_k(h) = \frac{1}{k!(n-1-k)!} * (g^{n-1-k} h^k).$$

Note that  $t_0(h) = g$  is the metric,  $t_{n-1}(h)$  coincides with the above defined adjugate of  $h$ . The terminology "higher adjugate" is justified by the following fact

$$t_k(h)(e_i, e_j) = \frac{1}{k!(n-1-k)!} (g^{n-1-k} h^k)(*e_i, *e_j).$$

The quantity  $\frac{1}{k!(n-1-k)!} (g^{n-1-k} h^k)(*e_i, *e_j)$  can be seen, like in the case  $k = n-1$  above, as  $(-1)^{i+j}$  multiplied by a higher determinant of the  $(n-1) \times (n-1)$  sub-matrix that is obtained from  $(h(e_i, e_j))$  by removing the  $i$ -th row and  $j$ -th column.

**Remark.** In view of formula 5 it is tempting to think that the "higher determinants"  $\frac{1}{k!(n-1-k)!} (g^{n-1-k} h^k)(*e_i, *e_j)$  coincide with the  $s_k$  invariant of the matrix. However it turns out that this not the case in general: The higher determinants are multilinear in the rows (or columns) of the matrix while the  $s_k$  invariants are not in general, as can one can see easily in the case of the trace  $s_1$ .

**1.2. Laplace expansions of the determinant and generalizations.** The following proposition provides a Laplace type expansion for all the  $s_k$  invariants of an arbitrary bilinear form.

**Proposition 1.4.** *For each  $k$ ,  $0 \leq k \leq n-1$ , we have*

$$(9) \quad (k+1)s_{k+1}(h) = \langle t_k(h), h \rangle.$$

Where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product of bilinear forms.

Before we prove this proposition let us clarify its relation to the classical Laplace expansion (called also cofactor expansion) of the determinant. Remark that for  $k = n - 1$  we have

$$ns_n = \langle t_{n-1}(h), h \rangle = \sum_{i,j=1}^n t_{n-1}(h)(e_i, e_j) h(e_i, e_j).$$

Recall that by definition the factor  $t_{n-1}(h)(e_i, e_j)$  is the usual  $(ij)$ -cofactor of the matrix  $(h(e_i, e_j))$  and  $s_n$  is its determinant. We therefore recover the classical Laplace expansion of the determinant.

**Remark.** Actually, Laplace expansion of the determinant is more refined. Precisely, for any  $i$  or  $j$  we have

$$s_n = \sum_{i=1}^n t_{n-1}(h)(e_i, e_j) h(e_i, e_j) = \sum_{j=1}^n t_{n-1}(h)(e_i, e_j) h(e_i, e_j).$$

*Proof.* Using basic properties of the exterior product of double forms and the generalized Hodge star operator, see [3, 5], it is straightforward that

$$\langle t_k(h), h \rangle = *(\{ *t_k(h) \} h) = * \left\{ \frac{g^{n-k-1}}{(n-k-1)!} \frac{h^{k+1}}{k!} \right\} = (k+1)s_{k+1}(h).$$

□

1.2.1. *Further Laplace expansions.* Recall that the determinant of  $h$  is determined by  $h^n$ . As the later expression can be written in several ways as a product  $h^{n-r} h^r$  for each  $r$ , we therefore get different expansions for the determinant by blocks as follows:

$$\begin{aligned} \det h &= * \frac{h^n}{n!} = * \frac{h^{n-r} h^r}{n!} = \frac{(n-r)! r!}{n!} \langle * \frac{h^{n-r}}{(n-r)!}, \frac{h^r}{r!} \rangle \\ &= \frac{1}{\binom{n}{r}} \sum_{\substack{i_1 < i_2 < \dots < i_r \\ j_1 < j_2 < \dots < j_r}} \epsilon(\rho) \epsilon(\sigma) \frac{h^r}{r!} (e_{i_1}, \dots, e_{i_r}, e_{j_1}, \dots, e_{j_r}) \frac{h^{n-r}}{(n-r)!} (e_{i_{p+1}}, \dots, e_{i_n}, e_{j_{p+1}}, \dots, e_{j_n}). \end{aligned}$$

Where  $\{e_1, e_2, \dots, e_n\}$  is an orthonormal basis of  $V$ ,  $\epsilon(\rho)$  and  $\epsilon(\sigma)$  are the signs of the permutations  $\rho = (i_1, \dots, i_n)$  and  $\sigma = (j_1, \dots, j_n)$  of  $(1, 2, \dots, n)$ . Recall that  $\frac{h^r}{r!} (e_{i_1}, \dots, e_{i_r}, e_{j_1}, \dots, e_{j_r})$  (resp.  $\frac{h^{n-r}}{(n-r)!} (e_{i_{p+1}}, \dots, e_{i_n}, e_{j_{p+1}}, \dots, e_{j_n})$ ) equals the determinant of the  $r \times r$  sub-matrix  $(h(e_{i_k}, e_{j_l}))$  for  $1 \leq k, l \leq r$  (resp the determinant of the  $n-r \times n-r$  sub-matrix  $(h(e_{i_k}, e_{j_l}))$  for  $p+1 \leq k, l \leq n$ ). Note that the second sub-matrix is just the co-matrix of the first sub-matrix, that is the sub-matrix obtained from the ambient matrix  $(h(e_i, e_j))$  of size  $n$  after removing the rows  $i_1, \dots, i_n$  and the columns  $j_1, \dots, j_n$ . Let us mention here also that the original Laplace expansion is finer

than the previous expansion, precisely it says that

$$\det h = \sum_{i_1 < i_2 < \dots < i_r} \epsilon(\rho)\epsilon(\sigma) \frac{h^r}{r!} (e_{i_1}, \dots, e_{i_r}, e_{j_1}, \dots, e_{j_r}) \frac{h^{n-r}}{(n-r)!} (e_{i_{p+1}}, \dots, e_{i_n}, e_{j_{p+1}}, \dots, e_{j_n}).$$

**Remark.** One can write easily similar expansions for all the lower  $s_k$  invariants of  $h$ . In fact, the product  $g^{n-k}h^k$  can be written in different ways as  $g^p h^q g^{n-k-p} h^{k-q}$  for  $0 \leq q \leq k$  and  $0 \leq p \leq n-k$ . Precisely we have, for  $1 \leq k \leq n$  and for every  $0 \leq q \leq k$  and  $0 \leq p \leq n-k$ , the following expansion

$$k!(n-k)!s_k(h) = *g^{n-k}h^k = *(g^p h^q g^{n-k-p} h^{k-q}) = \langle g^p h^q, *g^{n-k-p} h^{k-q} \rangle = \sum_{\substack{i_1 < i_2 < \dots < i_{p+q} \\ j_1 < j_2 < \dots < j_{p+q}}} \epsilon(\rho)\epsilon(\sigma) g^p h^q (e_{i_1}, \dots, e_{i_{p+q}}, e_{j_1}, \dots, e_{j_{p+q}}) g^{n-k-p} h^{k-q} (e_{i_{p+q+1}}, \dots, e_{i_n}, e_{j_{p+q+1}}, \dots, e_{j_n}).$$

Where as above  $\{e_1, e_2, \dots, e_n\}$  is an orthonormal basis of  $V$ ,  $\epsilon(\rho)$  and  $\epsilon(\sigma)$  are the signs of the permutations  $\rho = (i_1, \dots, i_n)$  and  $\sigma = (j_1, \dots, j_n)$  of  $(1, 2, \dots, n)$ .

### 1.3. Girard-Newton identities.

**Proposition 1.5** (Girard-Newton identities). *For  $0 \leq k \leq n-1$ , the trace of  $t_k$  is given by*

$$(10) \quad ct_k(h) = (n-k)s_k(h).$$

Where  $c$  denotes the contraction map.

*Proof.* Using basic properties of the exterior product of double forms and the the generalized Hodge star operator, see [3, 5], we immediately get

$$ct_k(h) = *g * t_k(h) = * \left\{ \frac{g^{n-k}}{(n-k-1)!} \frac{h^k}{k!} \right\} = (n-k)s_k(h).$$

□

In order to explain why the previous formula coincides with the classical Girard-Newton identities, we shall use lemma 1.8 which gives an explicit formula for the linear operator  $\overline{t_r(h)}$  that is associated to  $t_r(h)$  via the metric  $g$ . Denote by  $\bar{h}$  the linear operator associated to  $h$  and let  $p_i = \text{tr } \bar{h}^i$ . Lemma 1.8 shows that

$$\text{tr } \overline{t_r(h)} = \sum_{i=0}^r (-1)^i s_{r-i}(h) p_i.$$

Therefore we can reformulate the identity (10) as

$$\sum_{i=0}^r (-1)^i s_{r-i}(h) p_i = (n-r)s_r(h),$$

or

$$r s_r(h) = \sum_{i=1}^r (-1)^{i+1} s_{r-i}(h) p_i.$$

That are the celebrated classical Girard-Newton identities.

**Remark** (Terminology). The transformations  $t_r$  are famous in the literature as Newton's transformations. Up to the author's knowledge, it was Reilly [9] the first to call them as such (he treated only the case of diagonalizable matrices). He justifies this by the fact that they generate the classical Newton identities as above. With reference to the above discussion, the terminology adjugate (or cofactor) is in the author's opinion more appropriate.

**1.4. Higher adjugates and Laplace expansions.** Let  $h$  be a bilinear form on the  $n$ -dimensional Euclidean vector space  $(V, g)$ . We define, for  $0 \leq q \leq n$  and  $0 \leq r \leq n - q$ , the  $(r, q)$  adjugate of  $h$  denoted  $s_{(r,q)}(h)$  by

$$(11) \quad s_{(r,q)}(h) = \frac{1}{q!(n-q-r)!} * (g^{n-q-r} h^q).$$

Note that  $s_{(1,q)}(h) = t_q(h)$  is the adjugate of order  $q$  of  $h$  as defined in subsection 1.1 and  $s_{(0,q)}(h) = s_q(h)$  is the  $s_q$  invariant of  $h$ .

The higher adjugates  $s_{(r,q)}(h)$  of  $h$  satisfy similar properties like the usual adjugate of  $h$  which was discussed above. We list some of them in the following theorem

**Theorem 1.6.** *For any integers  $r$  and  $q$  such that  $0 \leq q \leq n$  and  $1 \leq r \leq n - q$  we have*

- *General Laplace's expansion:*

$$\frac{(q+r)!}{q!} s_{q+r}(h) = \langle s_{(r,q)}(h), h^r \rangle.$$

- *General Newton's identity:*

$$c(s_{(r,q)}(h)) = (n - q - r + 1) s_{(r-1,q)},$$

*Proof.* To prove the general Laplace's expansion we again use the formulas of [3] to get a one line proof as follows

$$\langle s_{(r,q)}(h), h^r \rangle = * \left( \frac{g^{n-q-r} h^q}{q!(n-q-r)!} h^r \right) = * \left( \frac{g^{n-q-r} h^{q+r}}{(q+r)!(n-q-r)!} \right) \frac{(q+r)!}{q!} = \frac{(q+r)!}{q!} s_{q+r}(h).$$

In the same way we prove the general Newton's identity as follows

$$\begin{aligned} c(s_{(r,q)}(h)) &= c * \left( \frac{g^{n-q-r} h^q}{q!(n-q-r)!} \right) = * g \left( \frac{g^{n-q-r} h^q}{q!(n-q-r)!} \right) \\ &= * \left( \frac{g^{n-q-r+1} h^q}{q!(n-q-r+1)!} \right) (n - q - r + 1) = (n - q - r + 1) s_{(r-1,q)}. \end{aligned}$$



□

**Remark.** (1) If the bilinear form  $h$  is diagonalizable, that is if there exists an orthonormal basis  $(e_i)$  of  $V$  such that  $h(e_i, e_j) = \lambda_i g(e_i, e_j)$  for all  $i, j$ . We call the real numbers  $\lambda_i$  the eigenvalues of  $h$ . Without loss of generality we assume that  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ .

It is not difficult to show that all the double forms  $s_{(r,q)}(h)$  with  $r \geq 1$  are then also diagonalizable in the sense that

$$s_{(r,q)}(h)(e_{i_1}, \dots, e_{i_r}, e_{j_1}, \dots, e_{j_r}) = \lambda_{i_1 i_2 \dots i_r} \frac{g^r}{r!}(e_{i_1}, \dots, e_{i_r}, e_{j_1}, \dots, e_{j_r}).$$

Where  $i_1 < i_2 < \dots < i_r$ ,  $j_1 < j_2 < \dots < j_r$  and the eigenvalues of  $s_{(r,q)}(h)$  are given by

$$\lambda_{i_1 i_2 \dots i_r} = \sum_{\substack{j_1 < j_2 < \dots < j_q \\ \{j_1, j_2, \dots, j_q\} \cap \{i_1, i_2, \dots, i_r\} = \emptyset}} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_q}.$$

- (2) In some applications it is useful to find the determinant of the sum of two matrices or more generally the  $s_k$  invariant of the sum. Using double forms formalism as above one can prove easily in one line the following identity for arbitrary bilinear forms  $A$  and  $B$  once seen as  $(1, 1)$  double forms and for  $1 \leq k \leq n$

$$s_k(A + B) = \sum_{i=0}^k \frac{1}{(k-i)!} \langle s_{(k-i,i)}(A), B^{k-i} \rangle.$$

### 1.5. Jacobi's formula.

**Proposition 1.7** (A Jacobi's formula for the  $s_k$  invariants). *Let  $h = h(t)$  be a one parameter family of bilinear forms on  $V$  then*

$$(12) \quad \frac{d}{dt} s_k(h) = \langle t_{k-1}(h), \frac{dh}{dt} \rangle.$$

*In particular, for  $k = n$  we recover the classical Jacobi's formula:*

$$(13) \quad \frac{d}{dt} \det(h) = \langle t_{n-1}(h), \frac{dh}{dt} \rangle.$$

*Proof.* Using basic properties of the exterior product of double forms and the generalized Hodge star operator, see [3, 5], we get

$$(14) \quad \begin{aligned} \frac{d}{dt} s_k(h(t)) &= \frac{d}{dt} \left( * \frac{g^{n-k} h^k(t)}{(n-k)!k!} \right) = * \left( \frac{g^{n-k} k h^{k-1}}{(n-k)!k!} \frac{dh}{dt} \right) \\ &= * \left( * \left( * \frac{g^{n-k} h^{k-1}}{(n-k)!(k-1)!} \right) \frac{dh}{dt} \right) = \langle t_{k-1}(h), \frac{dh}{dt} \rangle. \end{aligned}$$

□

**Remark.** If one allows the inner product  $g$  on  $V$  to vary as well, say  $g = g(t)$ , then at  $t = 0$  we have the following generalization of the previous formula:

$$(15) \quad \frac{d}{dt}s_k(h) = \langle t_{k-1}(h), \frac{dh}{dt} \rangle + \langle t_k - s_k g, \frac{dg}{dt} \rangle.$$

The proof is similar to the above one.

**1.6. Cayley-Hamilton Theorem.** It is now time to give a sense to the top  $t_k(h)$ , that is  $t_n(h)$ , where  $n$  is the dimension of the vector space  $V$ . To simplify the exposition we assume here in this section that  $h$  is a symmetric bilinear form. Recall that for  $1 \leq k \leq n-1$  we have, see [5]

$$(16) \quad t_k(h) = \frac{1}{k!(n-1-k)!} * (g^{n-1-k}h^k) = s_k g - \frac{1}{(k-1)!} c^{k-1} h^k.$$

It is then natural to define  $t_n(h)$  to be

$$(17) \quad t_n(h) = s_n(h)g - \frac{1}{(n-1)!} c^{n-1} h^n.$$

Recall that  $*h^n = n!s_n(h)$ , and therefore  $h^n = n!s_n(h) * 1 = s_n(h)g^n$ . Consequently we have

$$(18) \quad t_n(h) = s_n(h)g - s_n(h) \frac{1}{(n-1)!} c^{n-1} g^n = s_n(h)g - s_n(h)g = 0.$$

It turns out that the previous simple algebraic relation  $t_n(h) = 0$  is equivalent to the celebrated Cayley-Hamilton Theorem as we will see below.

**Lemma 1.8.** *Let  $0 \leq r \leq n-1$ , and  $\overline{t_r(h)}$  denotes the linear operator associated to  $t_r(h)$  then*

$$(19) \quad \overline{t_r(h)} = \sum_{i=0}^r (-1)^i s_{r-i}(h) \bar{h}^i.$$

Where  $\bar{h}$  is the linear operator that is asociated to  $h$ .

*Proof.* The previous lemma is a direct consequence of the following induction formula

$$(20) \quad \overline{t_r(h)} = s_r(h)I - \bar{h} \overline{t_{r-1}(h)}.$$

Where  $I$  is the identity operator. In order to prove the previous formula let us diagonalize  $\bar{h}$  and denote its eigenvalues by  $\lambda_1, \dots, \lambda_n$ . Then the eigenvalues of  $\overline{t_r(h)}$  are

given by (they are indexed by  $i$ )

$$\begin{aligned}
\sum_{\substack{i_1 < i_2 < \dots < i_r \\ i \notin \{i_1, i_2, \dots, i_r\}}} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_r} &= \frac{1}{r!} \sum_{\substack{i_1 \neq i_2 \neq \dots \neq i_r \\ i \notin \{i_1, i_2, \dots, i_r\}}} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_r} \\
&= \frac{1}{r!} \sum_{i_1 \neq i_2 \neq \dots \neq i_r} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_r} - \frac{1}{r!} \sum_{\substack{i_2 \neq \dots \neq i_r \\ i \notin \{i_2, \dots, i_r\}}} \lambda_i \lambda_{i_2} \dots \lambda_{i_r} - \dots \\
(21) \quad &= s_r(h) - \frac{r}{r!} \lambda_i \sum_{\substack{i_2 \neq \dots \neq i_r \\ i \notin \{i_2, \dots, i_r\}}} \lambda_{i_2} \dots \lambda_{i_r} \\
&= s_r(h) - \lambda_i \left( \sum_{\substack{i_2 < \dots < i_r \\ i \notin \{i_2, \dots, i_r\}}} \lambda_{i_2} \dots \lambda_{i_r} \right).
\end{aligned}$$

□

It results from the above discussion that

$$(22) \quad \overline{t_n(h)} = \sum_{i=0}^n (-1)^i s_{n-i}(h) \bar{h}^i.$$

and therefore the vanishing of  $t_n(h)$  is equivalent to the Cayley-Hamilton theorem:

$$(23) \quad \sum_{i=0}^n (-1)^i s_{n-i}(h) \bar{h}^i = 0.$$

1.6.1. *Cayley-Hamilton theorem vs. Infinitesimal Gauss-Bonnet theorem.* Let  $M$  be a compact smooth hypersurface of the Euclidean space of dimension  $2n+1$ . Denote by  $B$  the second fundamental form of  $M$  and by  $s_k(B)$  its  $s_k$  invariant. For each  $k$ ,  $0 \leq k \leq n$ , the first variation of the integral  $\int_M s_{2k}(B) \text{dvol}$  is up to a multiplicative constant, the integral scalar product  $\langle t_{2k}(B), B \rangle$ , where  $t_{2k}(B)$  is the adjugate transformation of  $B$  as above, see [5]. The later result can be seen as an integral Jacobi's formula. By Cayley-Hamilton theorem  $t_{2n}(B) = 0$  and therefore the integral  $\int_M s_{2n}(B) \text{dvol}$  does not depend on the geometry of the hypersurface. In fact, the previous integral is up to a multiplicative constant the Euler-Poincaré characteristic by the Gauss-Bonnet theorem. In this sense, Cayley-Hamilton theorem is indeed an infinitesimal Gauss-Bonnet theorem.

1.6.2. *Further algebraic identities of Cayley-Hamilton type.* We continue here to assume, as declared in the beginning of this subsection 1.6, that  $h$  is a symmetric bilinear form. In this case the higher adjugates  $s_{(r,q)}(h)$  are then as well symmetric  $(r, r)$  double forms.

Formula (15) of [3] provides the following expansion for  $0 \leq r \leq n$  and  $1 \leq q \leq n - r$ :

$$(24) \quad s_{(r,q)}(h) = \sum_{i=\max\{0,q-r\}}^q \frac{(-1)^{i+q}}{i!q!(i+r-q)!} g^{i+r-q} c^i h^q.$$

This new form of  $s_{(r,q)}(h)$  allows to extend its definition to the higher values of  $q$ , namely for  $q$  equal to  $n - r + 1, \dots, n$ . For instance in the top case  $q = n \geq 2$  we define

$$(25) \quad s_{(r,n)}(h) = \sum_{i=n-r}^n \frac{(-1)^{i+n}}{i!n!(i+r-n)!} g^{i+r-n} c^i h^n.$$

Recall that  $h^n = s_n(h)g^n$  and therefore  $c^i h^n = s_n(h) \frac{i!n!}{(n-i)!} g^{n-i}$ , consequently we have for  $r \geq 1$

$$s_{(r,n)}(h) = \sum_{i=n-r}^n \frac{(-1)^{i+n}}{(n-i)!(i+r-n)!} g^r = \frac{(-1)^r}{r!} \left( \sum_{j=0}^r (-1)^j \binom{r}{j} \right) g^r = 0.$$

We have therefore proved that

$$(26) \quad s_{(r,n)}(h) = 0 \text{ for all } 1 \leq r \leq n.$$

For  $r = 1$  we recover  $t_n(h) = 0$  that is the usual Cayley-Hamilton theorem for  $h$ . The next case is when  $q = n - 1 \geq 2$  and  $1 \leq r \leq n - 1$ , here we set

$$(27) \quad s_{(r,n-1)}(h) = \sum_{i=n-r-1}^{n-1} \frac{(-1)^{i+n-1}}{i!(n-1)!(i+r-n+1)!} g^{i+r-n+1} c^i h^{n-1}.$$

Next we are going to show that

$$(28) \quad s_{(r,n-1)}(h) = 0 \text{ for all } 2 \leq r \leq n - 1.$$

In order to prove the above identities, first remark that  $h^{n-1}$  is a  $(n-1, n-1)$  double form on an  $n$  dimensionnal vector space, then using proposition 2.1 of [8] we can write  $h^{n-1} = g^{n-2}k$  for some  $(1, 1)$  double form  $k$  on  $V$ . Consequently, using some identities from [3] we get for  $i \leq n - 2$  the following

$$\begin{aligned} c^i(h^{n-1}) &= c^i(g^{n-2}k) = *g^i * \frac{g^{n-2}k}{(n-2)!} (n-2)! \\ &= *g^i (-k + gck) (n-2)! = (- *g^i k + *g^{i+1}ck) (n-2)! \\ &= -i!(n-2)! \left( -\frac{g^{n-i-2}k}{(n-i-2)!} + \frac{g^{n-i-1}ck}{(n-i-1)!} \right) + \frac{(i+1)!n-2)!}{(n-i-1)!} g^{n-i-1}ck \\ &= \frac{(n-2)!i!}{(n-i-2)!} g^{n-i-2} \left( k + \frac{i(ck)}{n-i-1} g \right). \end{aligned}$$

For  $i = n - 1$ , we get

$$c^{n-1}h^n = ((n-1)!)^2 ck.$$

Consequently the formula above defining  $s_{(r,n-1)}(h)$  takes the form

$$\begin{aligned} s_{(r,n-1)}(h) &= \frac{ck}{r!}g^r + \sum_{i=n-1-r}^{n-2} \frac{(-1)^{i+n-1}}{(n-1)(i+r-n+1)!(n-i-2)!}g^{r-1} \left( k + \frac{i(ck)}{n-i-1}g \right) \\ &= \left( \sum_{i=n-1-r}^{n-2} \frac{(-1)^{i+n-1}}{(i+r-n+1)!(n-i-2)!} \right) \frac{g^{r-1}k}{n-1} + \left( \sum_{i=n-1-r}^{n-1} \frac{(-1)^{i+n-1}i}{(i+r-n+1)!(n-i-1)!} \right) \frac{g^r ck}{n-1}. \end{aligned}$$

Changing the index of both sums to  $j = i - n + 1 + r$  we immediately obtain

$$s_{(r,n-1)}(h) = \left( \sum_{j=0}^{r-1} \frac{(-1)^j}{j!(r-j-1)!} \right) \frac{(-1)^r g^{r-1}k}{n-1} + \left( \sum_{j=0}^r \frac{(-1)^j(j+n-1-r)}{j!(r-j)!} \right) \frac{(-1)^r g^r ck}{n-1}.$$

It is then easy to check that the previous two sums are both zero for  $r \geq 2$ .

In the same way we define  $s_{(r,n-i)}(h)$  using formula (24) and one can prove similarly, as in the cases where  $i = 0$  and  $i = 1$  above, the following general result

**Theorem 1.9** (A general Cayley-Hamilton theorem). *For  $1 \leq i+1 \leq r \leq n-i$  we have*

$$s_{(r,n-i)}(h) = 0.$$

In order to re-formulate the previous theorem in terms of operators, like in the usual form of Cayley-Hamilton theorem, one can use the following induction formula

$$\begin{aligned} (29) \quad s_{(r,q)}(h)(e_{i_1}, \dots, e_{i_r}, e_{i_1}, \dots, e_{i_r}) &= s_{(r-1,q)}(h)(e_{i_2}, \dots, e_{i_r}, e_{i_2}, \dots, e_{i_r}) \\ &\quad - h(e_{i_1}, e_{i_1})s_{(r,q-1)}(h)(e_{i_1}, \dots, e_{i_r}, e_{i_1}, \dots, e_{i_r}). \end{aligned}$$

Where  $\{e_1, \dots, e_n\}$  denotes an orthonormal basis diagonalizing  $h$  and  $i_1 \neq i_2 \neq \dots \neq i_r$ . Consequently, we obtain the following

$$(30) \quad \overline{s_{(r,q)}(h)}(e_{i_1}, \dots, e_{i_r}) = \sum_{k=0}^q (-1)^k \overline{s_{(r-1,q-k)}(h)}((e_{i_2}, \dots, e_{i_r}) \bar{h}^k(e_{i_1}),$$

where an over bar is made to indicate the corresponding associated linear operators.

In particular, we have for all  $1 \leq i \neq j \leq n$

$$(31) \quad \overline{s_{(2,q)}(h)}(e_i, e_j) = \sum_{k=0}^q (-1)^k \overline{t_{q-r}(h)}(e_j) \bar{h}^k(e_i).$$

Taking  $q = n - 1$ , the previous general Cayley-Hamilton theorem asserts that

**Corollary 1.10.** *Let  $V$  be a real vector space of finite dimension  $n \geq 3$  and let  $A$  be a (diagonalizable) linear operator on  $V$ . Denote by  $t_r(A) = \sum_{i=0}^r (-1)^i s_{r-i}(A)A^i$*

the  $r$ -th adjugate of  $A$  (that is the  $k$ -th Newton transformation), then we have for all  $1 \leq i \neq j \leq n$  the following Cayley-Hamilton theorem

$$(32) \quad \sum_{k=0}^{n-1} (-1)^k t_{q-r}(A)(e_j) A^k(e_i) = 0.$$

Where  $\{e_1, e_2, \dots, e_n\}$  denotes an orthonormal basis diagonalizing  $A$ .

**1.7. Skew-symmetric bilinear forms and the Pfaffian.** Let us suppose in this subsection that the bilinear form  $h$  is skew-symmetric. Then  $h$  can be seen either as a 2-form or as a  $(1, 1)$  double form like what we did up to here in this paper.

Suppose  $\dim V = n = 2k$  is even. We know already that  $*\frac{h^n}{n!}$  is the determinant of  $h$  once  $h$  is seen as a  $(1, 1)$  double form. However, if we perform the same operations on the 2-form  $h$  and of course with the ordinary exterior product of forms and the usual Hodge star operator we obtain a new invariant  $\text{Pf}(h)$  called the Pfaffian of  $h$ . Precisely, we have

$$(33) \quad \text{Pf}(h) = *\frac{h^k}{k!}.$$

It turns out that  $\text{Pf}(h)$  is a square root of the determinant of  $h$  that is  $(\text{Pf}(h))^2 = \det h$ . This identity can be quickly justified as follows:

We proceed by duality, in the case where  $h$  is a 2-form, then  $\frac{h^k}{k!} \otimes \frac{h^k}{k!}$  is an  $(n, n)$  double form. Since the space of  $(n, n)$  double forms on  $V$  is 1-dimensionnal vector space then it is proportionnal to the double  $*\frac{h^n}{n!}$  where in the last expression  $h$  is seen as a  $(1, 1)$  double form. It turns out that the previous two  $(n, n)$  double forms are equal. To show the desired equation it suffices to compare the image of the two double forms under the generalized Hodge star operator. From one hand we have

$$*\left(\frac{h^k}{k!} \otimes \frac{h^k}{k!}\right) = *\frac{h^k}{k!} \otimes *\frac{h^k}{k!} = (\text{Pf}(h))^2.$$

On the other hand we have

$$*\left(\frac{h^n}{n!}\right) = \det h.$$

The Pfaffian satisfies several similar properties to those of the determinant and one may use the exetrior product of exterior forms (as we did here for double forms) to prove such properties.

## 2. EUCLIDEAN INVARIANTS FOR SYMMETRIC $(2, 2)$ DOUBLE FORMS

In this section we are going to generalize the previous results to symmetric  $(2, 2)$  double forms that satisfy the first Bianchi identity. Recall that a  $(2, 2)$  symmetric double form is a multilinear form with four arguments that is skew symmetric with respect to the

interchange of the first two arguments or the last two, and it is symmetric if we interchange the first two arguments with the last two. The Riemann curvature tensor is a typical example.

During this section  $R$  denotes a symmetric  $(2, 2)$  double form on the  $n$ -dimensional Euclidean vector space  $(V, g)$  that satisfies the first Bianchi identity.

**2.1. The  $h_{2k}$  invariants vs  $s_k$  invariants.** For each  $k$ ,  $0 \leq 2k \leq \dim V = n$ , we define (by analogy to the  $s_k$  invariants of the previous section) the  $h_{2k}$  invariant of  $R$  to be

$$(34) \quad h_{2k}(R) = \frac{1}{(n-2k)!} * (g^{n-2k} R^k).$$

In particular  $h_0 = 1$  and  $h_n = *R^k$  in case  $n = 2k$ .

In the case where  $R$  is the Riemann curvature tensor of a Riemannian manifold,  $h_{2k}(R)$  is known as the  $2k$ -th Gauss-Bonnet curvature.

**Remark.** Suppose  $n = 2k$  is even and define the  $h_n$ -characteristic polynomial of  $R$  to be  $h_n \left( R - \lambda \frac{g^2}{2} \right)$ . A direct computation shows that

$$\begin{aligned} h_n \left( R - \lambda \frac{g^2}{2} \right) &= * \left( R - \lambda \frac{g^2}{2} \right)^k \\ &= * \sum_{i=0}^k \binom{k}{i} R^i \frac{(-1)^{k-i}}{2^{k-i}} \lambda^{k-i} g^{2k-2i} = \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{k-i}}{2^{k-i}} \lambda^{k-i} * (g^{2k-2i} R^i) \\ &= \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{k-i}}{2^{k-i}} (2k-2i)! h_{2i}(R) \lambda^{k-i}. \end{aligned}$$

The  $h_n$ -characteristic polynomial of  $R$  is therefore a polynomial of degree  $k$  in  $\lambda$  and its coefficients are all the  $h_{2i}(R)$  invariants of  $R$ , in particular these invariants coincide with the elementary symmetric functions in the roots of the  $h_n$ -characteristic equation of  $R$ . It would be therefore interesting to identify these roots and to relate them in particular to the eigenvalues of the linear operator associated to  $R$ .

**2.2. Adjugates of  $(2, 2)$  double forms.** Using the same procedure of cofactors as in the previous section we obtain several adjugates of  $R$ . Let us here examine the following

$$(35) \quad N_{2k}(R) = * \frac{g^{n-2k-2} R^k}{(n-2k-2)!} \text{ and } T_{2k}(R) = * \frac{g^{n-2k-1} R^k}{(n-2k-1)!}.$$

Note that  $N_{2k}(R)$  is defined for  $2 \leq 2k \leq n-2$  and it is a  $(2, 2)$  symmetric double form on  $V$  like  $R$  that satisfies the first Bianchi identity. On the other hand  $T_{2k}(R)$  is defined for  $2 \leq 2k \leq n-1$  and it is a symmetric bilinear form on  $V$ .

Theorem 4.1 of [3] provides explicit useful formulas for all the  $N_{2k}(R)$  and  $T_{2k}(R)$  as follows:

$$(36) \quad \begin{aligned} N_{2k}(R) &= \frac{c^{2k-2}R^k}{(2k-2)!} - \frac{c^{2k-1}R^k}{(2k-1)!}g + \frac{c^{2k}R^k}{2(2k)!}g^2, \\ T_{2k}(R) &= \frac{c^{2k}R^k}{(2k)!}g - \frac{c^{2k-1}R^k}{(2k-1)!}. \end{aligned}$$

In particular,  $T_2(R) = \frac{c^2R}{2}g - cR$  is the celebrated Einstein Tensor. The higher  $T_{2k}(R)$  are called Einstein-Lovelock tensors, see [6].

### 2.3. Laplace expansion of the $h_{2k}$ invariants and Avez formula.

**Theorem 2.1** (Laplace expansion of the  $h_{2k}$  invariants). *For  $4 \leq 2k+2 \leq n$  we have*

$$(37) \quad h_{2k+2}(R) = \langle N_{2k}(R), R \rangle.$$

*Proof.* Using the results of [3] one immediately has

$$\langle N_{2k}(R), R \rangle = * \left\{ \frac{g^{n-2k-2}R^{k+1}}{(n-2k-2)!} \right\} = \frac{c^{2k+2}R^{k+1}}{(2k+2)!}.$$

This completes the proof.  $\square$

As a consequence we recover Avez's formula for the second Gauss-Bonnet curvature as follows:

**Corollary 2.2** (Avez's Formula). *For  $n \geq 4$ , the second Gauss-Bonnet curvature equals*

$$h_4(R) = |R|^2 - |cR|^2 + \frac{1}{4}|c^2R|^2.$$

*Proof.* A direct application of formula 36 shows that

$$N_2(R) = R - (cR)g + \frac{c^2R}{4}g^2.$$

Consequently,

$$h_4(R) = \langle N_2(R), R \rangle = \langle R, R \rangle - \langle (cR)g, R \rangle + \langle \frac{c^2R}{4}g^2, R \rangle.$$

To complete the proof just recall that the contraction map  $c$  is the adjoint of the multiplication map by the  $g$ , see [3].  $\square$

In the same way one can prove easily the following generalization of Avez's formula, see [7],



**Corollary 2.3.** *For  $4 \leq 2k + 2 \leq n$ , the  $(2k + 2)$ -th Gauss-Bonnet curvature is determined by the last three contractions of  $R^k$  as follows:*

$$h_{2k+2} = \langle \frac{c^{2k-2}R^k}{(2k-2)!}, R \rangle - \langle \frac{c^{2k-1}R^k}{(2k-1)!}, cR \rangle + h_{2k}h_2.$$

#### 2.4. Girard-Newton identities.

**Theorem 2.4.** *Let  $c$  denotes the contraction map then we have*

$$cN_{2k}(R) = (n - 2k - 1)T_{2k} \text{ and } cT_{2k}(R) = (n - 2k)h_{2k}.$$

*Proof.* Using the identity  $c* = *g$  from [3], one easily gets the desired formulas as follows:

$$cN_{2k}(R) = c * \frac{g^{n-2k-2}R^k}{(n-2k-2)!} = * \frac{g^{n-2k-1}R^k}{(n-2k-1)!} \frac{(n-2k-1)!}{(n-2k-2)!} = (n-2k-1)T_{2k}.$$

Similarly,

$$cT_{2k}(R) = c * \frac{g^{n-2k-1}R^k}{(n-2k-1)!} = * \frac{g^{n-2k}R^k}{(n-2k)!} \frac{(n-2k)!}{(n-2k-1)!} = (n-2k)h_{2k}.$$

□

#### 2.5. Algebraic identities for $(2, 2)$ double forms.

2.5.1. *The case of even dimensions.* Suppose the dimension of the vector space  $V$  is even  $n = 2k$ . We shall now give a sense to the top  $T_{2k}(R)$ . Using formula 36 we naturally set

$$(38) \quad T_n(R) = \frac{c^n R^k}{n!} g - \frac{c^{n-1} R^k}{(n-1)!}.$$

**Proposition 2.5.** *Let  $R$  be a symmetric  $(2, 2)$  double form satisfying the first Bianchi identity on an Euclidean space of even dimension  $n$  then*

$$(39) \quad T_n(R) = 0.$$

*Proof.* Note first that if  $n = 2k$  then  $h_n = *R^k$  and therefore  $R^k = *h_n$ . Where  $*$  is the Hodge star operator acting on double forms. For any  $0 \leq r \leq n$  we have

$$c^r R^k = c^r * h_n = *g^r h_n = r! \frac{g^{n-r}}{(n-r)!} h_n.$$

That is

$$\frac{c^r R^k}{r!} = \frac{h_n}{(n-r)!} g^{n-r}.$$

Next using the definition of  $T_n$  above we easily get that

$$T_n = h_n g - h_n g = 0.$$

□

**Remark.** In this case where  $n = 2k$  one could using formula (36) define  $N_n(R)$  as well by setting

$$N_n(R) = \frac{c^{n-2}R^k}{(n-2)!} - \frac{c^{n-1}R^k}{(n-1)!}g + \frac{c^n R^k}{2(n!)}g^2.$$

A direct adaptation of the previous proof shows that we have the algebraic identity  $N_n(R) = 0$ .

**2.5.2. The case of odd dimensions.** Suppose now the dimension of the vector space  $V$  is odd say  $n = 2k + 1$ . We shall now give a sense the top  $N_{n-1}(R)$ . Note that the top  $T_{n-1}(R)$  is well defined and need not vanish in general. Using formula 36 we naturally set for  $n \geq 3$

$$(40) \quad N_{n-1}(R) = \frac{c^{n-3}R^k}{(n-3)!} - \frac{c^{n-2}R^k}{(n-2)!}g + \frac{c^{n-1}R^k}{2(n-1)!}g^2.$$

**Theorem 2.6.** *Let  $R$  be a symmetric  $(2, 2)$  double form satisfying the first Bianchi identity on an Euclidean space of odd dimension  $n \geq 3$  then*

$$(41) \quad N_{n-1}(R) = 0.$$

*Proof.* Let  $n = 2k + 1 \geq 3$ , note that  $R^k$  is a  $(n-1, n-1)$  double form on an  $n$  dimensionnal vector space, then using proposition 2.1 of [8] we can write  $R^k = g^{n-2}D$  for some  $(1, 1)$  double form  $D$  on  $V$ . Consequently, using some identities from [3] we get the following

$$\begin{aligned} c^{2k-2}R^k &= c^{n-3}(g^{n-2}D) = *g^{n-3} * g^{n-2}D \\ &= *g^{n-3}(n-2)!(-D + gD) = (n-2)!(- *g^{n-3}D + *g^{n-2}cD) \\ &= (n-2)!(n-3)! \left( gD + \frac{(n-3)cD}{2}g^2 \right). \end{aligned}$$

After contracting the previous identity twice we get

$$c^{2k-1}R^k = (n-2)!(n-2)!(D + (n-2)(cD)g), \text{ and } c^{2k}R^k = (n-1)!(n-1)!cD.$$

Consequently we have

$$\begin{aligned} N_{n-1}(R) &= \frac{c^{n-3}R^k}{(n-3)!} - \frac{c^{n-2}R^k}{(n-2)!}g + \frac{c^{n-1}R^k}{2(n-1)!}g^2 \\ &= (n-2)! \left( D + \frac{(n-3)cD}{2}g - D - (n-2)cDg + (n-1)\frac{cD}{2}g \right) g = 0. \end{aligned}$$

□

**Remark.** In dimension  $n = 3$  the previous theorem read

$$N_2(R) = R - (cR)g + \frac{c^2 R}{4}g^2 = 0.$$

In the context of Riemannian geometry where  $R$  represents the Riemann curvature tensor the previous identity is equivalent to the vanishing of the Weyl tensor in 3 dimensions, in fact in this dimension  $N_2(R)$  coincides with the Weyl tensor.

**2.5.3. Algebraic scalar identities for  $(2, 2)$  double forms.** Suppose the dimension  $n$  of our vector space  $V$  is odd, say  $n = 2k + 1$  and as above  $R$  is a symmetric  $(2, 2)$  double form that satisfies the first Bianchi identity. Corollary 2.3 allows one to define  $h_{n+1}(R) = h_{2k+2}(R)$ , precisely we set

$$h_{2k+2}(R) = \left\langle \frac{c^{2k-2}R^k}{(2k-2)!}, R \right\rangle - \left\langle \frac{c^{2k-1}R^k}{(2k-1)!}, cR \right\rangle + \left\langle \frac{c^{2k}R^k}{(2k)!}, \frac{c^2 R}{2} \right\rangle.$$

We are going to show that  $h_{2k+2}(R)$  as defined by the previous equation is zero.

We proceed as in the proof of Theorem 2.6 and using the same notations of that proof we have

$$\begin{aligned} \left\langle \frac{c^{2k-2}R^k}{(2k-2)!}, R \right\rangle &= (n-2)! \left\langle D + \frac{(n-3)cD}{2}g, R \right\rangle, \\ -\left\langle \frac{c^{2k-1}R^k}{(2k-1)!}, cR \right\rangle &= -(n-2)! \left\langle D + (n-2)cDg, cR \right\rangle, \\ \left\langle \frac{c^{2k}R^k}{(2k)!}, \frac{c^2 R}{2} \right\rangle &= (n-1)!cD \frac{c^2 R}{2}. \end{aligned}$$

Taking the sum of the above three equation we immediately prove the vanishing of  $h_{2k+2}(R)$ . Thus we have proved the following scalar identities

**Proposition 2.7.** *Let  $R$  be a symmetric  $(2, 2)$  double form satisfying the first Bianchi identity on an Euclidean vector space of odd dimension  $n = 2k + 1 \geq 3$  then*

$$\left\langle \frac{c^{n-3}R^k}{(n-3)!}, R \right\rangle - \left\langle \frac{c^{n-2}R^k}{(n-2)!}, cR \right\rangle + \left\langle \frac{c^{n-1}R^k}{(n-1)!}, \frac{c^2 R}{2} \right\rangle = 0.$$

*In particular, for  $n = 3$  we have*

$$\langle R, R \rangle - \langle cR, cR \rangle + \frac{1}{4} (c^2 R)^2 = 0.$$

**Remark.** In the context of Riemannian geometry, where  $R$  is the Riemann curvature tensor (seen as a  $(2, 2)$  double form), the previous scalar curvature identities coincide with Gilkey-Park-Sekigawa universal curvature identities [2] which are shown to be unique. Also the identities of Proposition 2.5 coincide with the symmetric 2-form valued universal curvature identities of [2] where they are also shown to be unique. The higher algebraic identities, that are under study here in this paper, can be seen

then as symmetric double form valued universal curvature identities in the frame of Riemannian geometry.

**2.5.4. Higher algebraic identities for  $(2, 2)$  double forms.** Let  $k \geq 1$  and  $0 \leq r \leq n - 2k$  and let  $R$  as above be a symmetric  $(2, 2)$  double form on the  $n$ -dimensional Euclidean vector space  $(V, g)$  that satisfies the first Bianchi identity. We define the  $(r, 2k)$ -adjugate of  $R$ , denoted  $h_{(r, 2k)}(R)$ , by the following formula

$$(42) \quad h_{(r, 2k)}(R) = \frac{1}{(n - 2k - r)!} * (g^{n-2k-r} R^k).$$

Note that  $h_{(r, 2k)}(R)$  for  $r = 0$  (resp.  $r = 1$ ,  $r = 2$ ) coincides with  $h_{2k}(R)$  (resp.  $T_{2k}(R)$ ,  $N_{2k}(R)$ ).

Theorem 4.1 of [3] shows that

$$(43) \quad h_{(r, 2k)}(R) = \sum_{i=\max\{0, 2k-r\}}^{2k} \frac{(-1)^i}{i!(r - 2k + i)!} g^{r-2k+i} c^i R^k.$$

This last formula (43) allows us to define  $h_{(r, 2k)}(R)$  for higher  $r$ 's that is for  $r > n - 2k$ . The following theorem which provides general identities and generalize Proposition 2.5 and Theorem 2.6 can be proved in the same way

**Theorem 2.8.** *Let  $R$  be a symmetric  $(2, 2)$  double form satisfying the first Bianchi identity on an Euclidean vector space of dimension  $n$ .*

(1) *If  $n = 2k$  is even then*

$$h_{(r, n-2i)}(R) = 0 \text{ for } 2i + 1 \leq r \leq n - 2i.$$

(2) *If  $n = 2k + 1$  is odd then*

$$h_{(r, n-2i-1)}(R) = 0 \text{ for } 2i + 2 \leq r \leq n - 2i - 1.$$

Remark that we recover Proposition 2.5 for  $n = 2k, r = 1, i = 0$  and Theorem 2.6 is obtained for  $n = 2k + 1, r = 2, i = 0$ .

## 2.6. Jacobi's formula for double forms.

**Proposition 2.9** (Jacobi's formula). *Let  $R = R(t)$  be a one parameter family of  $(2, 2)$  double forms then*

$$(44) \quad \frac{d}{dt} h_{2k}(R) = \langle k N_{2k-2}(R), \frac{dR}{dt} \rangle.$$

*Proof.*

$$\begin{aligned}
 (45) \quad \frac{d}{dt} h_{2k}(R) &= \frac{d}{dt} \left( * \frac{g^{n-2k} R^k(t)}{(n-2k)!} \right) = * \left( \frac{g^{n-2k} k R^{k-1}}{(n-2k)!} \frac{dR}{dt} \right) \\
 &= * \left( * \left( * \frac{g^{n-2k} k R^{k-1}}{(n-2k)!} \right) \frac{dR}{dt} \right) = \langle k N_{k-1}(R), \frac{dR}{dt} \rangle.
 \end{aligned}$$

□

**Remark.** If one allows the scalar product to vary as well say  $g = g(t)$  the previous formula takes the following form at  $t = 0$ :

$$(46) \quad \frac{d}{dt} h_{2k}(R) = \langle k N_{2k-2}(R), \frac{dR}{dt} \rangle + \langle T_{2k} - h_{2k}g, \frac{dg}{dt} \rangle.$$

The proof is similar to the above one.

**2.7. Algebraic identities vs. infinitesimal Gauss-Bonnet theorem.** Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n = 2k$ . Denote by  $R$  its Riemann curvature tensor seen here as a  $(2, 2)$  double form and, and let  $h_{2r}(R)$  be the corresponding Gauss-Bonnet curvatures as above. For each  $r$ ,  $0 \leq r \leq n$ , the gradient of the Riemannian functional  $\int_M h_{2r}(R) \text{dvol}$  at  $g$ , once restricted to metrics of unit volume, is equal to  $T_{2r}(R)$ , where  $T_{2r}(R)$  is  $h_{2r}(R)$  adjugate of  $R$  as above, it is known in geometry as the Einstein-Lovelock tensor, see [6]. The later result can be seen as an integral Jacobi's formula. Consequently, the algebraic identity  $T_n(R) = 0$  shows that the integral  $\int_M h_n(R) \text{dvol}$  does not depend on the metric  $g$  of  $M$ . In fact, the previous integral is up to a multiplicative constant the Euler-Poincaré characterestic by the Gauss-Bonnet theorem.

Here again, as in the situation of section 1.6.1, the linearized version of the Gauss-Bonnet theorem is an algebraic identity for  $(2, 2)$  double forms.

**2.8. Pfaffian of 4-forms.** Let  $\omega$  be a 4-form on the Euclidean vector space  $V$ . Remark that  $\omega$  can be naturally considered as a symmetric  $(2, 2)$  double form.

Suppose  $\dim V = n = 4k$  is a multiple of 4. By definition we have  $\frac{1}{(2k)!} * \omega^{2k} = h_n(\omega)$  is the  $h_n$  invariant of  $\omega$  once it is considered as a  $(2, 2)$  double form. However, if we perform the same operations on the 4-form  $\omega$  and of course with the ordinary exterior product of forms and the usual Hodge star operator we obtain a new invariant  $\text{Pf}(\omega)$ , which we shall call the Pfaffian of  $\omega$ . Precisely, we set

$$(47) \quad \text{Pf}(\omega) = * \frac{\omega^k}{k!}.$$

Using the same argument as in section 1.7 it is plausible that  $(\text{Pf}(\omega))^2 = h_n(\omega)$ . It would be interesting to investigate the properties of this invariant for 4-forms.

### 3. HIGHER ADJUGATE TRANSFORMATIONS AND APPLICATIONS

We shall now in this section generalize the previous results to higher symmetric  $(p, p)$ -double forms.

Let  $\omega$  be a symmetric  $(p, p)$ -double form satisfying the first Bianchi identity, we define its adjugate transformation of order  $(r, pq)$  to be

$$(48) \quad h_{(r,pq)}(\omega) = * \frac{g^{n-pq-r} \omega^q}{(n-pq-r)!}.$$

Where  $0 \leq r \leq n-pq$ . The result is a symmetric  $(r, r)$ -double form. We remark that for  $\omega = h$  a  $(1, 1)$  symmetric double form that is a symmetric bilinear form we recover the invariants of section 1 as follows

$$h_{(0,q)}(h) = q!s_q(h), \quad h_{(1,q)}(h) = q!t_q(h) \text{ and } h_{(r,q)}(h) = q!s_{(r,q)}(h).$$

Furthermore, for  $\omega = R$  a symmetric  $(2, 2)$  double form, we recover the invariants of section 2 as follows

$$h_{(0,2q)}(R) = h_{2q}(R), \quad h_{(1,2q)}(R) = T_{2q}(R) \text{ and } h_{(2,2q)}(R) = N_{2q}(R).$$

One can generalize without difficulties the results of the previous sections to this general setting. First using Theorem 4.2 of [3] one easily gets

$$(49) \quad h_{(r,pq)}(\omega) = \sum_{i=\max\{0,pq-r\}}^{pq} \frac{(-1)^{i+pq}}{i!(r-pq+i)!} g^{r-pq+i} c^i \omega^q.$$

As a first result we have the following Laplace type expansion:

**Theorem 3.1.** *Let  $\omega$  be a symmetric  $(p, p)$ -double form satisfying the first Bianchi identity on an  $n$ -dimensional Euclidean vector space  $V$ , Let  $q$  be a positive integer such that  $n \geq 2pq$  then*

$$(50) \quad \frac{c^{2pq}(\omega^{2q})}{(2pq)!} = \langle h_{(pq,pq)}(\omega), \omega^q \rangle = \sum_{r=0}^{pq} \frac{(-1)^{r+pq}}{(r!)^2} \langle c^r \omega^q, c^r \omega^q \rangle.$$

*Proof.* From one hand we have

$$(51) \quad \langle h_{(pq,pq)}(\omega), \omega^q \rangle = \langle * \frac{g^{n-2pq} \omega^q}{(n-2pq)!}, \omega^q \rangle = * \left( \frac{g^{n-2pq} \omega^{2q}}{(n-2pq)!} \right) = \frac{c^{2pq}(\omega^{2q})}{(2pq)!}.$$

On the other hand we have

$$(52) \quad * \left( \frac{g^{n-2pq} \omega^q}{(n-2pq)!} \right) = \sum_{r=0}^{pq} \frac{(-1)^{r+pq}}{r!} \frac{g^r c^r \omega^q}{r!}.$$

Consequently it is straightforward that

$$(53) \quad \frac{c^{2pq}(\omega^{2q})}{(2pq)!} = \sum_{r=0}^{pq} \frac{(-1)^{r+pq}}{(r!)^2} \langle g^r c^r \omega^q, \omega^q \rangle = \sum_{r=0}^{pq} \frac{(-1)^{r+pq}}{(r!)^2} \langle c^r \omega^q, c^r \omega^q \rangle.$$

□

Taking  $\omega = R$  a  $(2, 2)$  double form we get

**Corollary 3.2** (General Avez Formula). *Let  $R$  be a symmetric  $(2, 2)$ -double form satisfying the first Bianchi identity on an  $n$ -dimensional Euclidean vector space  $V$ , Let  $q$  be a positive integer such that  $n \geq 4q$  then*

$$(54) \quad h_{4q}(R) = \sum_{r=0}^{2q} \frac{(-1)^r}{(r!)^2} \langle c^r R^q, c^r R^q \rangle.$$

In particular,

$$(55) \quad h_4 = |R|^2 - |cR|^2 + \frac{1}{4}|c^2 R|^2 \text{ and } h_8 = |R^2|^2 - |cR^2|^2 + \frac{1}{4}|c^2 R^2|^2 - \frac{1}{36}|c^3 R^2|^2 + \frac{1}{576}|c^4 R^2|^2.$$

**Remark.** The previous corollary improves a similar result of [3] obtained for the case  $n = 4q$ .

**Corollary 3.3.** *Let  $h$  be a symmetric bilinear form on an  $n$ -dimensional Euclidean vector space  $V$ , Let  $q$  be a positive integer such that  $n \geq 2q$  then*

$$(56) \quad s_{2q}(h) = \sum_{r=0}^q \frac{(-1)^{r+q}}{(r!)^2} \langle c^r h^q, c^r h^q \rangle.$$

In particular,

$$(57) \quad s_2(h) = -|h|^2 + |ch|^2 \text{ and } s_4(h) = |h^2|^2 - |ch^2|^2 + \frac{1}{4}|c^2 h^2|^2.$$

The following proposition can be seen as a generalization of the classical Newton identities

**Proposition 3.4.** *For  $1 \leq r \leq n - pq$  we have*

$$c(h_{(r,pq)}(\omega)) = (n - pq - r + 1)h_{(r-1,pq)}(\omega)$$

.

*Proof.* Using the identity  $c* = *g$  of [3], one easily gets the desired formulas as follows:

$$c(h_{(r,pq)}(\omega)) = c * \frac{g^{n-pq-r}\omega^k}{(n-pq-r)!} = * \frac{g^{n-pq-r+1}R^k}{(n-pq-r)!} = (n-pq-r+1)h_{(r-1,pq)}(\omega).$$

□

We finish this section by establishing higher algebraic identities. First note that formula (49) allows to define  $h_{(r,2k)}(\omega)$  for higher  $r$ 's that is in the cases where  $r > n - pk$ . The following theorem provides general algebraic identities and generalize Theorems 2.8 and 2.6 and Proposition 2.5. It can be proved by imitating the proof of Theorem 2.6.

**Theorem 3.5.** *Let  $\omega$  be a symmetric  $(p, p)$  double form satisfying the first Bianchi identity on an Euclidean vector space of dimension  $n$ . Then*

$$h_{(r, pk-pi)}(\omega) = 0, \text{ for } n - pk + pi + 1 \leq r \leq pk - pi.$$

*In particular, we have*

(1) *If  $n = pk$  is a multiple of  $p$  then*

$$h_{(r, n-pi)}(\omega) = 0 \text{ for } pi + 1 \leq r \leq n - pi.$$

(2) *If  $n = pk + 1$  is then*

$$h_{(r, n-pi-1)}(\omega) = 0 \text{ for } pi + 2 \leq r \leq n - pi - 1.$$

Remark that we recover the results of Theorem 2.8 for  $p = 2$ , Proposition 2.5 for  $n = 2k, r = 1, i = 0$  and Theorem 2.6 is obtained for  $n = 2k + 1, r = 2, i = 0$ .

#### 4. FINAL REMARKS AND OPEN QUESTION

**4.1. Adjugate transformations vs. Gilkey's restriction map.** Following [1, 2] we define  $\mathcal{I}_{m,n}^{p+1}$  to be the space of invariant local formulas for symmetric  $(p, p)$  double forms that satisfy the first Bianchi identity and that are homogeneous of degree  $n$  in the derivatives of the metric and which are defined in the category of  $m$  dimensional Riemannian manifolds. In particular  $\mathcal{I}_{m,n}^1 = \mathcal{I}_{m,n}$  is the space of scalar invariant local formulas and  $\mathcal{I}_{m,n}^2$  is the space of symmetric 2-form valued invariants.

Recall that the homogeneity of order  $n$  for  $\omega = \omega(g) \in \mathcal{I}_{m,n}^{p+1}$  is equivalent to

$$\omega(c^2 g) = \frac{1}{c^{n-2p}} \omega(g),$$

for all  $c \neq 0$ .

The last property implies in particular that if  $\omega(g) \in \mathcal{I}_{m,n}^{p+1}$  then its full contraction  $c^p \omega(g) \in \mathcal{I}_{m,n}$  and  $c^{p-1} \omega(g) \in \mathcal{I}_{m,n}^2$ .

The Gilkey's restriction map  $r : \mathcal{I}_{m,n} \rightarrow \mathcal{I}_{m-1,n}$  is closely related to the adjugate transformations as we will explain below.

Let  $\omega(g) \in \mathcal{I}_{m,n}^{p+1}$ , recall that the  $(1, pq)$  adjugate transformation of  $\omega$  is  $h_{(1,pq)}(\omega) = * \frac{g^{n-pq-1} \omega^q}{(n-pq-1)!} \in \mathcal{I}_{m,n}^2$ . For a tangent vector  $v$ , we have

$$h_{(1,pq)}(\omega)(v, v) = \frac{g^{(n-1)-pq} \omega^q}{((n-1)-pq)!} (*v, *v).$$

That is the restriction of the invariant formula  $Q = * \frac{g^{n-pq} \omega}{(n-pq)!} = \frac{c^{pq} \omega^q}{(pq)!} \in \mathcal{I}_{m,n}$  to the  $(n-1)$  dimensional orthogonal complement of the vector  $v$ .

More generally,  $h_{(r,pq)}(\omega)$  is given by  $r$  successive applications of Gilkey's restriction map to the invariant formula  $\frac{c^{pq} \omega}{(pq)!} \in \mathcal{I}_{m,n}$ .



**4.2. Spectrum of adjugate transformations of  $(2, 2)$  double forms.** Let  $R$  be a symmetric  $(2, 2)$  double form defined over the  $n$ -dimensional Euclidean vector space  $(V, g)$ . Denote by  $\lambda_1, \lambda_2, \dots, \lambda_N$ , where  $N = \frac{n(n-1)}{2}$ , the eigenvalues of the linear operator  $\Lambda^2 V \rightarrow \Lambda^2 V$  that is canonically associated to  $R$ . For each  $k$  with  $1 \leq k \leq n-2$ , the exterior product  $g^k R$  is a  $(k+2, k+2)$  symmetric double form and therefore has real eigenvalues once considered as an operator  $\Lambda^{k+2} V \rightarrow \Lambda^{k+2} V$ . The eigenvalues of the operator are expected to be polynomials in the eigenvalues of  $R$ . The question here is to find explicit formulas for the eigenvalues of  $g^k R$  in terms of  $\lambda_1, \lambda_2, \dots, \lambda_N$ .

In the case  $k = n-2$  the answer is trivial as we have in this case one single eigenvalue and it is equal to  $\sum_{i=1}^N \lambda_i$ .

To motivate this question let us recall that the Weitzenböck transformation of order  $p$ ,  $2 \leq p \leq n-2$ , of the double form  $R$  is given by [4]

$$\mathcal{N}_p(R) = \left( \frac{gcR}{p-1} - 2R \right) \frac{g^{p-2}}{(p-2)!}.$$

The positivity of the transformation  $\mathcal{N}_p(R)$  has important consequences in Riemannian geometry via the celebrated Weitzenböck formula.

The true question is then to determine the eigenvalues of  $\mathcal{N}_p(R)$  in terms of the eigenvalues of  $R$ .

More generally, what are the eigenvalues of the exterior products  $g^p R^q$  and the contractions  $c^p R^q$  in terms of  $\lambda_1, \lambda_2, \dots, \lambda_N$ ?

In particular, find explicit formulas for the invariants  $h_{2k}(R)$ , for  $k > 1$ , in terms of  $\lambda_1, \lambda_2, \dots, \lambda_N$ .

**4.3. Geometric applications of the general algebraic identities.** We have seen in the sequel of the paper that the identity  $t_n(h) = 0$  for a bilinear form leads to a linearized version of the Gauss-Bonnet theorem for compact hypersurfaces of the Euclidean space. Furthermore, for a  $(2, 2)$  double form  $R$ , the identity  $T_n(R) = 0$ , for  $n$  even and in the context of Riemannian manifolds, is an infinitesimal form of the general Gauss-Bonnet-Chern theorem for compact manifolds.

It is then natural to ask what differential or topological consequences can be drawn from the identity  $N_{n-1}(R)$  for a compact Riemannian manifold of odd dimension  $n$ ?

Note that for a 3-dimensionnal Riemannian manifold, the identity  $N_{n-1}(R) = 0$  for the Riemann curvature tensor  $R$  is equivalent to the vanishing of the Weyl tensor.

The same question can be asked for the other different higher algebraic identities established here in this paper.

**4.4. A Pfaffian for  $2k$  forms.** Let  $\omega$  be a  $2k$ -form on an Euclidean vector space  $(V, g)$  of finite dimension  $n$ . Remark that  $\omega$  can be naturally considered as a  $(k, k)$  double form.

Suppose  $\dim V = n = 2kq$  is a multiple of  $2k$ . By definition we have  $*\frac{\omega^{2q}}{(2q)!} = h_{(0,n)}(\omega)$  is the  $h_{(0,n)}$  invariant of  $\omega$  once it is considered as a  $(k, k)$  double form. However, if we

perform the same operations on the  $2k$ -form  $\omega$  and of course with the ordinary exterior product of forms and the usual Hodge star operator we obtain another invariant  $\text{Pf}(\omega)$ , which we shall call the Pfaffian of  $\omega$ . Precisely, we set

$$(58) \quad \text{Pf}(\omega) = * \frac{\omega^q}{q!}.$$

Using the same simple argument as in section 1.7 it is plausible that  $(\text{Pf}(\omega))^2 = h_{(0,n)}(\omega)$ . It would be interesting to investigate the properties of this new invariant for  $2k$ -forms.

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